An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

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Background

- High order numerics (5th order and higher) has become more common in research, and is beginning to appear in industrial uses of CFD.
- Overset gridding and overset solver techniques is one of several enabling technologies to evaluating flow field around complex (and moving) geometries.
- Use of overset is still a tool for “experts” (or researchers), but is slowly penetrating into industry.

Purpose of the Presentation

- Ignite discussion of the often overlooked issue of interpolation accuracy and its effects with high order solvers
- Present interesting findings through simple, and not so simple, computational examples
Limitations of this Study

• CFD on block structured overset meshes
  (for use in FV/FD codes)

• Two solvers:
  ➢ In-house Penn State code (PSU)
  ➢ OVERFLOW 2.2 (OF)

• Overset domain connectivity determined using Suggar++
  and read into the solvers using DiRTLib

• Explicit isoparametric Lagrangian interpolation method to
  determine donor weights
For generally spaced source pts:

\[ x_0 \quad x_1 \quad x_2 \quad \hat{x}_\xi \quad x_3 \quad x_4 \quad x_5 \]

Example: N=6

Standard Lagrangian Interpolation (SLI)

\[
f(\hat{x}) = \sum_{i=0}^{N-1} P_i(\hat{x})f(x_i)
\]

\[
P_i(\hat{x}) = \prod_{j=0, j\neq i}^{N-1} \frac{(\hat{x} - x_j)}{(x_i - x_j)}
\]

If source pts are equally spaced, \( \Delta x = \text{constant} \):

\[
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\]

\[ \delta \]

Example: N=6

Isoparametric Lagrangian Interpolation (ILI)

\[
f(\delta) = \sum_{i=0}^{N-1} R_i(\delta)f(x_i)
\]

\[
R_i(\delta) = \frac{(-1)^{N+i-1}}{[N-(i+1)]!i!} \prod_{l=0, l\neq i}^{N-1} (\delta - l)
\]
Isoparametric Lagrangian Interpolation (ILI)

For non-equally spaced source points.

\[
f(\delta) = \sum_{i=0}^{N-1} R_i(\delta) f(x_i)
\]

\[
R_i(\delta) = \frac{(-1)^{N+i-1}}{[N - (i + 1)]! i!} \prod_{l=0,l\neq i}^{N-1} (\delta - l)
\]

where \( \delta \) is found by minimizing the functional:

\[
F(x_i, \hat{x}, \delta) = \sum_{i=0}^{N-1} R_i(\delta)x_i - \hat{x} = 0
\]

But what about loss of accuracy???
Example: Interpolate a cubic function using uniform source grid
Example: Interpolate a cubic function using uniform source grid
Example: Interpolate a cubic function using quadratic-ly stretched source grid
Example: Interpolate a cubic function using quadratically stretched source grid
Examination of 1D interpolation error using different function types, donor stencils, and interpolation methods.

<table>
<thead>
<tr>
<th>Function Number</th>
<th>Equation</th>
<th>Uniform spaced source:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( f(x) = 10x )</td>
<td>N2</td>
</tr>
<tr>
<td>2</td>
<td>( f(x) = 4x^2 + 10x )</td>
<td>N4</td>
</tr>
<tr>
<td>3</td>
<td>( f(x) = 0.2x^3 + 4x^2 + 10x )</td>
<td>N6</td>
</tr>
<tr>
<td>4</td>
<td>( f(x) = 0.6x^4 + 0.2x^3 + 4x^2 + 10x )</td>
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<tr>
<td>5</td>
<td>( f(x) = 0.8x^5 + 0.6x^4 + 0.2x^3 + 4x^2 + 10x )</td>
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<tr>
<td>6</td>
<td>( f(x) = 5e^{-x^2/6} )</td>
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<td>7</td>
<td>( f(x) = 5\sin x/x )</td>
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</table>

Standard Lagrangian (SLI)

\[
f(\hat{x}) = \sum_{i=0}^{N-1} P_i(\hat{x}) f(x_i)\]

\[
P_i(\hat{x}) = \prod_{j=0, j\neq i}^{N-1} \frac{(\hat{x} - x_j)}{(x_i - x_j)}\]

Isoparametric Lagrangian (ILI)

\[
f(\delta) = \sum_{i=0}^{N-1} R_i(\delta) f(x_i)\]

\[
R_i(\delta) = \frac{(-1)^{N+i-1}}{[N - (i + 1)]!} \frac{1}{i!} \prod_{l=0, l \neq i}^{N-1} (\delta - l)\]
Interpolation error from uniform source grid

<table>
<thead>
<tr>
<th>Basis Function Number</th>
<th>Stencil size, N</th>
<th>Isoparametric Lagrangian</th>
<th>Standard Lagrangian</th>
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Some observations:

For polynomial functions:
- $N$ should be $> \text{function order}$

For non-polynomial functions:
- Error reduces with higher $N$
Interpolation error from quadratic source grid

<table>
<thead>
<tr>
<th>Basis Function Number</th>
<th>Stencil size, N</th>
<th>Maximum Difference Error:</th>
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</tr>
</tbody>
</table>

Some observations:

For polynomial functions:
• Error reduces with higher N

For non-polynomial functions:
• Error reduces with higher N
• ILI sometimes better than SLI

What kind of function is a flow variable (i.e u,v,w,p,e,ρu,ρv)?
1-D Told Us ... 

• If your function is a polynomial, then polynomial interpolation works great! (Duh)

• If you are using Isoparametric Lagrangian Interpolation (ILI), then you should use a stencil wider than the order of the function.

• If the function is not a polynomial, then usually wider the stencil the better.

• When using ILI for flow variables, all this is interpreted as: 
  *Use an overset stencil width (N) wider than the order of the scheme.*

All these points will apply to 3-D
Solvers

• Penn State (PSU) in-house code:
  ➢ Density based finite volume
  ➢ Roe flux based upwind scheme
  ➢ Up to 7\textsuperscript{th} order accurate inviscid flux differencing
  ➢ Tailored for incompressible flows via preconditioning
  ➢ Overset via DiRTlib, also supports block-to-block

• OVERFLOW 2.2 (OF)
  ➢ Density based finite difference
  ➢ Multiple spatial and time integration schemes
  ➢ Up to 6\textsuperscript{th} order accurate inviscid flux differencing
  ➢ Widely used for compressible flows, also has low Mach preconditioning
  ➢ Overset built-in (or XINTOUT), \textit{but has been modified to use DiRTlib in order to read high-order DCI files}
**DiRTlib/Suggar++**

- DiRTlib is a library that can be linked to OVERFLOW (or any equipped solver) that encapsulates overset interpolation and communication to the solver by making calls to solver interface routines that are specific to the solver.
  - Gets data from solver memory for use in donor interpolations
  - Puts data into solver memory for use at fringes
  - Fills solver IBLANK array to define holes and fringes

- Suggar++ is a general overset grid assembly code that provides a domain connectivity information (DCI) file to DiRTlib.

*Noack, Boger*
Relevant Features of DiRTlib/Suggar++

Suggar++
- Supports the specification of an *arbitrary* number of fringe layers as is required for the solver to maintain its high-order spatial discretization
- Can provide standard and high order (Lagrangian) weights

DiRTlib
- Supports arbitrary number of fringe layers
- Makes no assumption on the number of weights
Standard Lagrangian Interpolation (SLI)
In Three Dimensions

\[ f(\tilde{x}) = \sum_{k=0}^{N_\zeta-1} \sum_{j=0}^{N_\eta-1} \sum_{i=0}^{N_\xi-1} P_{ijk}(\tilde{x})f(\bar{x}_{ijk}) \]

\[ P_{ijk}(\tilde{x}) = P_i(\tilde{x})P_j(\tilde{y})P_k(\tilde{z}) \]

\[ = \prod_{l=0, l \neq i}^{N_\xi-1} \frac{(\tilde{x} - x_l)}{(x_i - x_l)} \prod_{m=0, m \neq j}^{N_\eta-1} \frac{(\tilde{y} - y_m)}{(y_j - y_m)} \prod_{n=0, n \neq k}^{N_\zeta-1} \frac{(\tilde{z} - z_n)}{(z_k - z_n)} \]

Note: Depending on orientation of the source points, \( P \) may be undefined.
Isoparametric Lagrangian Interpolation (ILI)
In Three Dimensions

\[ f(\bar{x}) = \sum_{k=0}^{N\xi-1} \sum_{j=0}^{N\eta-1} \sum_{i=0}^{N\xi-1} R_{ijk}^{\xi}(\delta_{\xi})R_{ijk}^{\eta}(\delta_{\eta})R_{ijk}^{\zeta}(\delta_{\zeta}) f(\bar{x}) \]

\[ R_i(\delta) = \alpha_i \prod_{j=0, j \neq i}^{N-1} (\delta - j) \quad \alpha_i = \frac{(-1)^{N+i-1}}{[N - (i + 1)]! i!} \]

where \( \delta \) is found by minimizing the functionals:

\[ F(\bar{x}, \hat{x}, \bar{\delta}) = \sum_{k=0}^{N\xi-1} \sum_{j=0}^{N\eta-1} \sum_{i=0}^{N\xi-1} R_{ijk}^{\xi}(\delta_{\xi})R_{ijk}^{\eta}(\delta_{\eta})R_{ijk}^{\zeta}(\delta_{\zeta}) \bar{x} - \hat{x} = 0 \]

This is the high order Lagrangian technique implemented in Suggar++.
Convecting Inviscid Vortex

(PSU: incompressible; OF: isentropic, M=0.2)

Description:
• Prescribe an initial inviscid vortex subject to a uniform cross flow
• Domain: Uniform background grid, with a series of different inset grids
• Boundary Conditions: Free stream, periodic BCs in flow direction
• Numerics: 3rd, 5th order accurate with N2, N4, and N6 overset interpolation

Purpose:
• Examine differences in accuracy (vortex dissipation and location) and between standard and high-order interpolation.
An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

2D Convecting Inviscid Vortex, Code: PSU
Vortex Dissipation, Uniform Inset Grid, 3rd Order

![Graphs showing vortex dissipation for different grid resolutions and interpolation accuracies.](image)
An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

2D Convecting Inviscid Vortex, Code: PSU
Vortex Dissipation, Quadratic Inset Grid, 5th Order
An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

2D Conveeting Inviscid Vortex, Code: OF Vortex Dissipation, Uniform Inset Grid, 5th Order

![Graphs showing the effects of overset interpolation accuracy in a high-order CFD solver.](image)
2D Convecting Inviscid Vortex, Code: PSU
Vortex Location, Quadratic Inset Grid, 3rd Order
2D Convecting Inviscid Vortex, Code: OF
Vortex Location, Uniform Inset Grid, 5th Order
2D Cylinder in Cross Flow With Overset Patch, Code:OF

Description:
• Canonical Karman vortex shedding flow (laminar)
• Domain: Uniform background grid with and without oblique hole, covered by non-uniform overset patch grid. 3 fringe layers
• Numerics: WENOM, 2nd order time(10 Newton sub its), ADI P-C

Purpose:
• Compare the vortex street downstream of the patch, with the baseline vortex street without the hole/patch

Re_D = 150
M = 0.2
Laminar
Conclusions:

- Grid obliqueness and stretching accentuates the need to use high-order interpolation with 3 fringe layers to preserve WENOM accuracy.
3D Cylindrical Column in Cross Flow, Codes: PSU, OF

Description:
• Column standing on solid surface

Purpose:
• Examine changes in shedding frequency due to overset treatment

Re_D = 150
PSU: Incompress
OF: M = 0.2
Laminar
An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

3D Cylindrical Column in Cross Flow

Intentionally introduced extraneous overset grids in order to create multiple overset boundaries
Frequency of oscillating lateral (v) velocity:

5th Order, Overset Stencil=6 (O5N6): 0.1389 Hz
5th Order, Overset Stencil=2 (O5N2): 0.1417 Hz

A 2% change with only several overset grids!
3D Rotor Hub, Code: PSU

Spinning notional scaled rotor tested in Water Tunnel at PSU. Free stream = 6.5 m/s

- Interest in flow structures at locations downstream where empennage would be
3D Rotor Hub Close-up

- Upper Spider
- Main Hub Arms
- Lower Spider
- Scissors
- Swashplate
3D Rotor Hub Test

Flow

Near Wake

Far Wake

LDV PIV

LDV

LDV

SPIV

≈ \(2 R_{\text{Hub}}\)

≈ \(4 R_{\text{Hub}}\)

≈ \(7 R_{\text{Hub}}\)
3D Rotor Hub CFD

Rotor in tunnel simulated using PSU and OF codes
- 99 Structured overset blocks
- 3rd order upwind bias
- Overset interpolation using N2 and N4
Wake Oscillating Vertical Velocity Comparisons

Far Wake

<table>
<thead>
<tr>
<th>w</th>
<th>(m/s)</th>
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<td>Pt 5</td>
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- PIV Data
- PSU CFD, O3N2 Overset
- PSU CFD, O3N4 Overset
An Examination of the Effects of Overset Interpolation Accuracy in the Context of a High-Order CFD Solver

High-Order Overset is Great, But it ....

• Requires more overlap in the grid system (to keep donor qual=1)
  ➢ For 3rd order upwind bias, standard (N2) overset needs:
    2 fringe layers, overlap=5
  ➢ For 5th order (e.g. WENO), high-order (N6) overset needs:
    3 fringe layers, overlap=9!!!!
• Requires more computer time (solver and Suggar++)
• Can be susceptible to under/over shoots (just like any polynomial interpolation can/will do)
Summary

• Using simple cases, it was determined that:
  - High-order interpolation better preserves flow structures across overset boundaries
  - More fringe layers (3) with standard interpolation is not sufficient. High-order interpolation and more fringe layers is required to preserve accuracy of high-order numerics

• With increasing order of accuracy of the interpolation ($N$), a larger donor stencil is needed, which in turn requires more overlap in the overset gridding and more runtime requirements
Thank you!